

EE228 – Solving Differential Equations – DePiero

This handout provides a brief description of the process for solving linear differential equations having constant coefficients. The technique presented is the ‘method of undetermined coefficients’. We proceed by finding the ‘Zero Input Response’ (ZIR) and then by finding the ‘Zero State Response’ (ZSR). These two responses are then added to yield the complete response. The single input case is addressed, along with possible initial conditions (I.C.). Note that the system dynamics (as expressed by the Natural Response) contribute to both ZIR and ZSR.

The n^{th} order differential equation of the form:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_{n-1} y^{(1)}(t) + a_n y(t) = x(t)$$

has a characteristic equation

$$s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0$$

1. Find ZIR, the zero-input response, $y_{zi}(t)$
 - a. Find the roots of the characteristic equation: s_i .
 - b. The form of $y_{zi}(t)$ is determined by the s_i roots. This is the part of the natural response, $y_{N1}(t)$, due to I.C. only (not due to the forcing function). See Table 1 to determine form of $y_{N1}(t)$.
 - c. Use given I.C. to find unknown coefficients of $y_{zi}(t)$.
2. Find the forced response, $y_F(t)$
 - a. Form of $y_F(t)$ is determined by the forcing function. See Table 2.
 - b. Substitute $y_F(t)$ into the differential equation, including $x(t)$ on RHS, and solve for unknown coefficients in $y_F(t)$.
3. Find ZSR, the zero-state response, $y_{zs}(t)$
 - a. The form of $y_{zs}(t) = y_F(t) + y_{N2}(t)$. Use $y_F(t)$ as found in step 2b. Include the generic form of $y_{N2}(t)$, having unknown coefficients.
 - b. Solve for unknown coefficients of $y_{N2}(t)$ using zero I.C.
4. Find total response $y(t) = y_{zi}(t) + y_{zs}(t)$.

Note in step 1, $y_{N1}(t)$ describes the component of the natural response associated with the I.C. In step 3 $y_{N2}(t)$ describes the component of the natural response associated with the forcing function.

Also, when terms such as $A x(t) + B dx(t)/dt$ appear on the RHS, it may be helpful to first find $y_{zs}(t)$ due to just the $x(t)$ forcing function. Then use linearity and superposition to find the overall ZSR due to all the terms on the RHS. This overall ZSR may then be added to $y_{zi}(t)$ to yield $y(t)$. Discontinuities in RHS introduce impulses, due to derivatives with $u(t)$ for example.

Root of Characteristic Equation	Form of Natural Response
Real and Distinct: r	$K e^{rt}$
Complex Conjugate: $b + jw$	$e^{bt} [K_1 \cos(wt) + K_2 \sin(wt)]$
Real, Repeated: r^{p+1}	$e^{rt} (K_0 + K_1 t + K_2 t^2 + \dots + K_p t^p)$

Table 1. Form of Natural Response for LTI Systems

Forcing Function (RHS)	Form of Forced Response
C_0	C_1
e^{at}	$C e^{at}$
$\cos(wt + p_1)$	$C_1 \cos(wt) + C_2 \sin(wt)$ or $C \cos(wt + p_2)$
$e^{at} \cos(wt + p_1)$	$e^{at} [C_1 \cos(wt) + C_2 \sin(wt)]$
t^p	$C_0 + C_1 t + C_2 t^2 + \dots + C_p t^p$

Table 2. Form of Forced Response for LTI Systems. For entries including e^{at} forcing function (*), if ‘a’ is also a root of the characteristic equation, repeated r times, then forced response should be multiplied by t^r .