EE328 – Formal Solution of Difference Equations – DePiero

This handout provides a brief description of the process for solving linear difference equations having constant coefficients. The technique presented is the 'method of undetermined coefficients', and is similar to methods employed for differential equations. We proceed by finding the 'Zero Input Response' (ZIR) and then by finding the 'Zero State Response (ZSR). These two responses are then added to yield the complete response. Note that the system dynamics (as expressed by the natural response) contribute to both ZIR and ZSR. The Q^{th} order difference equation of the form:

$$y[n] + A_1 y[n-1] + \dots + A_Q y[n-Q] = x[n]$$

has a characteristic equation

$$1 + A_1 z^{-1} + A_2 z^{-2} + \dots + A_Q z^{-Q} = z^Q + A_1 z^{Q-1} + \dots + A_Q = 0$$

with Q roots. The roots determine the form of the transient, and are identical to the roots of the denominator of H(z). The form of the forced (or steady state) response matches the form of the forcing function.

1. Find the forced response $y_F[n]$ and the <u>form of</u> the natural response $y_N[n]$.
   a. Form of $y_N[n]$ is determined via roots of the characteristic eq. See Table 1.
   b. Form of $y_F[n]$ matches the form of the forcing function, x[n]. See Table 2.
   c. Find C_i scale factors of $y_F[n]$ by substituting into the given difference eq.
   $$y_F[n] + A_1 y_F[n-1] + \dots + A_Q y_F[n-Q] = x[n]$$
   Use x[n] on RHS, without any additional B_0 scaling or other B_k x[n-k] terms.

2. Find ZSR, the zero-state response $y_{zs}[n]$, by first finding $y_0[n]$.
   a. The form of $y_0[n] = y_F[n] + y_{N2}[n]$. Use $y_F[n]$ as found in step 1b. Include the generic form of $y_{N2}[n]$, having unknown coefficients.
   b. Solve for unknown coefficients of $y_{N2}[n]$ using zero I.C, as with $y_0[-1] = 0$…
   c. Find $y_{zs}[n]$ via linearity and superposition:
   $$y_{zs}[n] = B_0\, y_0[n] + B_1\, y_0[n-1] + \dots + B_P\, y_0[n-P]$$

3. Find ZIR, the zero-input response, $y_{zi}[n]$
   a. The form of $y_{zi}[n] = y_{N2}[n]$, where $y_{N2}[n]$ has unknown coefficients.
   b. Evaluate unknown coefficients, using given I.C., similar to step 2b.

4. Find total response $y[n] = y_{zi}[n] + y_{zs}[n]$.

Root of Characteristic Equation	Form of Natural Response
Real and Distinct: r	$K r^n$
Complex Conjugate: $r e^{jw}$	$r^n [K_1 \cos(wt) + K_2 \sin(wt)]$
Real, Repeated: r^{p+1}	$r^n (K_0 + K_1 n + K_2 n^2 + \dots + K_p n^p)$

Table 1. Form of Natural Response for LTI systems

Forcing Function (RHS)		Form of Forced Response
C_0		C_1
a^n	*	$C\ a^n$
$\cos(wt + p_1)$		$C_1 \cos(wt) + C_2 \sin(wt)$ *or* $C \cos(wt + p_2)$
$r^n \cos(wt + p_1)$	*	$r^n [C_1 \cos(wt) + C_2 \sin(wt)]$
n^p		$C_0 + C_1 n + C_2 n^2 + \dots + C_p n^p$

Table 2. Form of Forced Response for discrete LTI systems. For entries including a^n forcing function (*), if 'a' is also a root of the characteristic eq, repeated q times, then forced response should be multiplied by n^q.

Example

Given: $y[n] = 4\,x[n] + x[n-1] - (1/3)\,y[n-1]$, $x[n] = 2\,u[n]$, and that $y[1-] = 1$
Find y[n], and then compare result with that obtained via the recursive method.

Following steps outlined in prior procedure.

1a) Finding characteristic equation

$$y[n] + (1/3)\,y[n-1] = 0$$

$$1 + (1/3)\,z^{-1} = 0 \quad \text{or} \quad z + (1/3) = 0$$

Hence a root $(z = -1/3)$ is present. Via Table 1, $y_N[n] = K\,(-1/3)^n$, $n>=0$

1b) The forcing function $x[n] = 2\,u[n]$. Hence the form of $y_F[n] = C_0$

1c) Substituting $y_F[n]$ into the difference equation,

$$y[n] + (1/3)\,y[n-1] = x[n]$$

and introducing the forcing function yields

$$y_F[n] + (1/3)\,y_F[n-1] = 2\,u[n]$$

which simplifies to

$$C_0 + (1/3)\,C_0 = 2, \quad \text{for large n.}$$

Thus $C_0 = 3/2$.

2a) The form of $y_0[n] = y_F[n] + y_{N2}[n] = 3/2 + K\,(-1/3)^n$.

2b) Finding K using zero IC

$$y_0[-1] = 0 = 3/2 + K\,(-1/3)^n, \quad \text{for } n = -1$$

$$-3/2 = K\,(-3), \quad \text{or } K = ½.$$

2c) Applying superposition, associated with an actual forcing function $4\,x[n] + x[n-1]$

$$y_{ZS}[n] = 4\,y_0[n] + y_0[n-1] = 4\,\{3/2 + ½\,(-1/3)^n\} + 3/2 + ½\,(-1/3)^{n-1}$$

$$y_{ZS}[n] = 7.5 + ½\,(-1/3)^n$$

3a) The form of $y_{Zi}[n] = K\,(-1/3)^n$ based on Table 1.
3b) Using the IC $y_{Zi}[-1] = 1$, yields

$$1 = K\,(-1/3)^{-1} \quad \text{or} \quad 1 = -3K. \quad \text{Hence } K = -1/3.$$

4) Find complete solution, $y[n] = y_{Zi}[n] + y_{ZS}[n]$, which simplifies to

$$y[n] = 7.5 + (1/6)\,(-1/3)^n \quad \text{for } n >= 0$$

Comparing to the recursive method

n	x[n]	x[n-1]	$y[n] = 4\,x[n] + x[n-1] - (1/3)\,y[n-1]$	$y[n] = 7.5 + (1/6)\,(-1/3)^n$
-1	0	0	1	1
0	2	0	7.67	7.67
1	2	2	7.44	7.44