EE 525
More Notes on Multiple Random Variables
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Estimating a Mean and Covariance Matrix from Sample Data

Sample values of measurements, as with $X = [\alpha \quad \beta]^T$ for example, may be described by estimating the mean, M, and covariance matrix, C, via

$$M \approx \frac{1}{N}\sum_{i=1}^{N} X_i \qquad C \approx \frac{1}{N}\sum_{i=1}^{N}(X_i - M)(X_i - M)^T$$

The mean may be thought of as describing a typical measurement, and the covariance matrix as a measure of the spread of measurements about the mean. For (2x1) measurements with zero mean the covariance matrix has the form

$$C = \begin{bmatrix} E[\alpha^2] & E[\alpha\beta] \\ E[\alpha\beta] & E[\beta^2] \end{bmatrix} = \begin{bmatrix} \sigma_\alpha^2 & \rho\sigma_\alpha\sigma_\beta \\ \rho\sigma_\alpha\sigma_\beta & \sigma_\beta^2 \end{bmatrix}$$

Where ρ is the normalized correlation coefficient. By definition $\rho = E[\alpha\beta]/(\sigma_\alpha\sigma_\beta)$. This results in $-1 \le \rho \le 1$.

Bivariate Gaussian Probability Density Function

If the above measurements $X = [\alpha \quad \beta]^T$ have a Gaussian, or 'normal', distribution then a bivariate Gaussian PDF may be used to describe how likely a given pair of (α, β) values are. For a mean M, covariance matrix C, and (2x1) measurements (n=2), the PDF has a form

$$f(X) = \frac{1}{(2\pi)^{n/2} \, |C|^{1/2}} e^{-\frac{1}{2}[(X-M)^T C^{-1}(X-M)]} = \frac{1}{(2\pi)^{n/2} \, |C|^{1/2}} e^{-\frac{d(X)^2}{2}}$$

Where d(X) corresponds to the Mahalanobis distance for a measurement X.

The covariance matrix is symmetric $C = C^T$ and positive definite. For a given constant d (a Mahalanobis distance) the set of all values of x such that

$$(x - M)^T C^{-1}(x - M) = d^2$$

form an ellipse. This ellipse is centered at the mean, M. For 2x1 measurements the direction of the major and minor axes of the ellipse are determined by the eigenvectors of C and the half-axis lengths are given by the square root of the associated eigenvalue. The eigenvectors and eigenvalues provide a simple way to determine the shape of the ellipse, which determines where measurements are likely to fall. (For d=3, ~99.7% of the measurements are expected to fall within the ellipse).

Probability Density Function for a Transformed Random Variable

Given a random variable, λ, with PDF: $f_\lambda(\lambda)$. A new random variable, defined by the transformation $v = g(\lambda)$ will have a PDF: $f_v(v) = \left|\frac{d\lambda}{dv}\right| f_\lambda(h(v))$ where h(v) is the inverse transformation $\lambda = h(v)$.