

EE 525
Notes on Random Processes
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Random Process

A random process is a signal that takes on values, which are determined (at least in part) by chance. A sinusoid with amplitude that is given by a random variable is an example of a random process. A random process cannot be predicted precisely. However, a deterministic signal is completely predictable.

An ergodic process is one in which time averages may be used to replace ensemble averages. As signals are often functions of time in signal processing applications, ergodicity is a useful property. An example of an ensemble average is the mean win at the blackjack tables across a whole casino, in one day. A similar time average could be the mean win at a particular blackjack table over every day for a month, for example. If these averages were approximately the same, then the process of blackjack winning would appear to be ergodic. Ergodicity can be difficult to prove or demonstrate, hence it is often simply assumed.

Probabilistic Description of a Random Process

Although random processes are governed by chance, more typical values and trends in the signal value can be described. A probability density function can be used to describe the typical intensities of the random signal (over all time). An autocorrelation function describes how similar the signal values are expected to be at two successive time instances. It can distinguish between 'erratic' signals versus a 'lazy random walk'.

Autocorrelation

$$R_x(t_1, t_2) = E[X(t_1)X(t_2)]$$

If the autocorrelation for a process depends only on the time difference $t_2 - t_1$, then the random process is deemed 'wide sense stationary'. For a WSS process:

$$R_x(\tau) = E[X(t)X(t + \tau)]$$

For a WSS, ergodic, process autocorrelation may be evaluated via a time average

$$R_x(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t)X(t + \tau) dt \quad (\text{Autocorrelation function, for continuous-time})$$

$$R_x(m) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N X(k)X(k + m) \quad (\text{Autocorrelation sequence, for discrete-time})$$

Note the $1/T$ factor is necessary for non-periodic power signals or for random signals (which are assumed to exist for all time). FYI when working with deterministic signals, the definition for the autocorrelation integral is changed. For example with a periodic signal, T may be set to the period. For a finite energy signal the $1/T$ may be dropped, making the definition more similar to the convolution integral.

Autocorrelation properties of a stationary process:

- $R_x(0)$ is the mean-square value of the process. (Compare to definition of $E[X^2]$).
- $R_x(\tau)$ is an even function.
- $|R_x(\tau)| \leq R_x(0)$ for all τ .
- If $X(t)$ is periodic then $R_x(\tau)$ is also periodic.

Crosscorrelation Function for a Stationary Process

$$R_{xy}(\tau) = E[X(t)Y(t + \tau)]$$

- Properties: $R_{xy}(\tau) = R_{yx}(-\tau)$. Also peak doesn't necessarily occur at $\tau = 0$.

Power Spectral Density

For a stationary process, the Wiener-Khinchine relation states:

$$S_x(j\omega) = \mathfrak{F}[R_x(\tau)]$$

Where $\mathfrak{F}[\cdot]$ denotes Fourier transform. $S_x(j\omega)$ can be thought of as the 'expected power' of the random process. This is because

$$S_x(j\omega) = \lim_{T \rightarrow \infty} E \left[\frac{1}{T} |\mathfrak{F}[X_T(t)]|^2 \right]$$

Where $X_T(t)$ is a truncated version of the random process. The quantity $\frac{1}{T} |\mathfrak{F}[X_T(t)]|^2$ is known as the periodogram of $X(t)$. The PSD may be estimated computationally, by taking the average of many examples of the power spectrum, over a relatively long intervals of the process $X(t)$. This reveals the expected power density at various frequencies. (One such interval of $X(t)$ would not typically have the power density of the time averaged version).

- Properties: $S_x(j\omega)$ is real, symmetric and nonnegative.

Cross Spectral Density for a Stationary Process

$$S_{xy}(j\omega) = \mathfrak{F}[R_{xy}(\tau)]$$

- Properties: $S_{xy}(j\omega) = S_{yx}^*(j\omega)$. Cross spectral densities are not typically real.

Common Random Processes and Sequences

Gauss-Markov Random Process (Continuous)

- Probability Density Function: Gaussian, with zero mean and variance σ^2 .
- $R_x(\tau) = \sigma^2 e^{-\beta|\tau|}$ $S_x(j\omega) = 2\sigma^2 \beta / (\omega^2 + \beta^2)$
- Stationary and physically realizable.
- Note: Particularly useful because it is simple and provides a reasonably accurate model for a wide range of physical processes.

Gaussian-White Sequence (Discrete)

- Probability Density Function: Gaussian, with zero mean and variance σ^2 .
- $R_x(m) = \sigma^2 \delta(m)$ Samples of (discrete) spectrum have magnitude σ^2 . Spectrum is band-limited due to the Fs sample rate.
- Stationary and physically realizable.

Gaussian-White Process (Continuous)

- Probability Density Function: Gaussian, with zero mean and infinite variance.
- $R_x(\tau) = A\delta(\tau)$ $S_x(j\omega) = A$
- Stationary and **not** physically realizable. (However still useful in analyses).
- Signal changes "infinitely far, infinitely fast"

Band-limited White Process (Continuous)

- Probability Density Function: Unspecified – can vary. $\sigma^2 = 2WA$
- $R_x(\tau) = 2WA \sin(2\pi W\tau) / (2\pi W\tau)$ $S_x(j\omega)$ is a rectangular pulse with amplitude A and bandwidth $2\pi W$ (W is in Hertz)
- Stationary and physically realizable.

Weiner Process (“Random Walk” or “Brownian Motion”) (Continuous)

Defined as $X(t) = \int_0^t n(u)du$ where $n()$ is a Gaussian-White Process.

- *Non-stationary*, with $E[X(t=t_1)] = 0$ and $E[X^2(t=t_1)] = t_1$.
- Probability Density Function: Gaussian, with zero mean and $\sigma^2 = t$.
- Autocorrelation Function: $R_x(t_1, t_2) = \min(t_1, t_2)$

Estimating Autocorrelation from Real Data

An autocorrelation sequence may be estimated from N samples of a discrete-time process $X(k)$ via

$$R_x(m) \approx V_x(m) = \frac{1}{N} \sum_{k=1}^N X(k)X(k+m)$$

This is an unbiased estimate. The variance of an estimate $V_x(\tau)$ for a continuous-time process $X_T(t)$ of duration T is, in general,

$$\text{Variance}\{V_x(\tau)\} = \frac{4}{T} \int_0^\infty R_x^2(\tau) d\tau$$

For a Gauss-Markov Process this is

$$\text{Variance}\{V_x(\tau)\} = \frac{2\sigma^4}{\beta T}$$

Description of a Random Process Using a Multivariable Probability Density Function

Another way to describe changes in a random process is to use a multivariable PDF with values of the process, $X(t)$, at successive time instances. For example the PDF for $f_r(\tau)$ could be defined for a Gauss-Markov process with

$$r = \begin{bmatrix} x_0 = X(t) \\ x_1 = X(t + \Delta T) \\ x_2 = X(t + 2\Delta T) \end{bmatrix}$$

A multivariable Gaussian PDF then describes how likely a given triplet of values are. The PDF would have a covariance matrix

$$C = \begin{bmatrix} E[x_0^2] & E[x_0 x_1] & E[x_0 x_2] \\ E[x_1 x_0] & E[x_1^2] & E[x_1 x_2] \\ E[x_2 x_0] & E[x_2 x_1] & E[x_2^2] \end{bmatrix}$$

The above E[] may be computed directly from the autocorrelation function of the Gauss-Markov process. The PDF in the above example is $f_{x_1, x_2, x_3}(x_1, x_2, x_3)$. As with any Gaussian process, higher order PDFs such as this may be found directly from the process model.