EE 525
Notes on Random Variables
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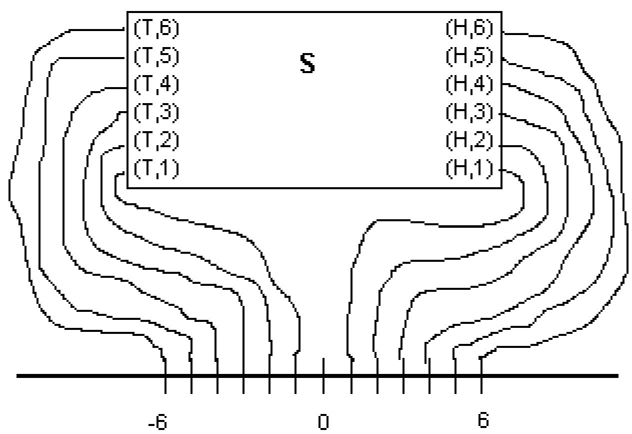
Random Variable
A random variable X can be considered a function that maps all elements of the sample space into points on the real number line or some parts thereof.

Example 1:
An experiment consists of rolling a die and flipping a coin. The random variable is a function X chosen such that:

(1) a coin head (H) corresponds to positive values of X that are equal to the number shown on the die and
(2) a coin tail (T) corresponds to negative values of X that are equal to the number shown on the die



Example 2:
A pointer on a wheel is spun. The possible outcomes are the numbers 0 to 12 marked on the wheel. The sample space consists of the numbers in the set $\{0 < s \le 12\}$. The random variable (r.v.) is defined by: $X = X(s) = s^2$.

An r.v. is a single valued function and the set $\{X \le x\}$ is an event for any real number.

$$P\{X = -\infty\} = 0\,,\quad P\{X = \infty\} = 0$$

Example 1 illustrates a discrete r.v. and example 2 illustrates a continuous r.v.

Examples 1 and 2 illustrate the freedom provided in the mapping between the "objects" in a chance experiment (such as coins and dice) and the values of a random variable, on the real number line.

Distribution Function

$P\{X \le x\}$ is the probability of the event $\{X \le x\}$. We call this function the "cumulative probability distribution function" of the r.v. *X* and it is denoted by $F_X(x)$.

$$F_X(x) = P\{X \le x\}$$

Properties:

1. $F_X(-\infty) = 0$
2. $F_X(\infty) = 1$
3. $0 \le F_X(x) \le 1$
4. $F_X(x_1) \le F_X(x_2)$ if $x_1 < x_2$
5. $P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$
6. $F_X(x^-) = F_X(x)$

If *X* is a discrete r.v., then

$$F_X(x) = \sum_{i=1}^{N} P\{X = x_i\} u(x - x_i)$$

where $u(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ and $P(x_i) = P\{X = x_i\}$

$$\text{i.e., } F_X(x) = \sum_{i=1}^{N} P(x_i) u(x - x_i)$$

Density Function

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$f_X(x)$ is the "density function" of the r.v. *X*. For discrete r.v.'s $\delta(x)$ is used to describe the derivative of $F_X(x)$. $\delta(x)$ is described by its integral property:

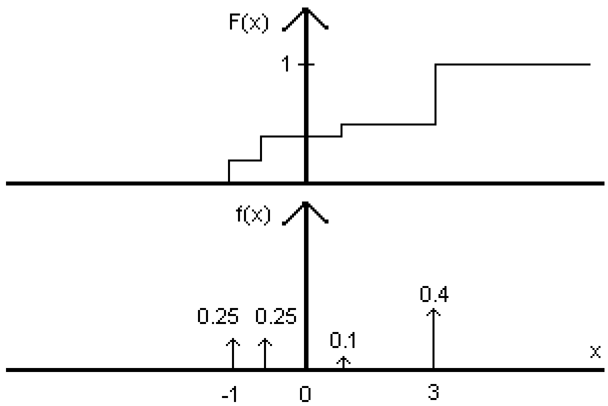
$$\Phi(x_0) = \int_{-\infty}^{\infty} \Phi(x) \delta(x - x_0)$$

where $\Phi(x)$ is a function continuous at $x = x_0$.

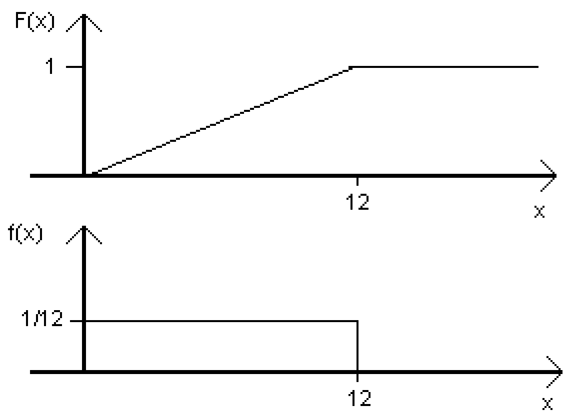
$$\delta(x) = \frac{du(x)}{dx} \text{ or } \int_{-\infty}^{x} \delta(\zeta) d\zeta = u(x)$$

$$f_X(x) = \sum_{i=1}^{N} P(x_i) \delta(x - x_i)$$

Example:
A discrete r.v.



A continuous r.v.



Properties of $f_X(x)$:

1. $0 \le f_X(x) \ \ \forall x$
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
3. $F_X(x) = \int_{-\infty}^{x} f_X(\zeta)d\zeta$
4. $P\{x_1 < x < x_2\} = \int_{x_1}^{x_2} f_X(x)dx$

Expected Value:
Expected value of an r.v. X is defined by:

$$E[X] = \overline{X} = \int_{-\infty}^{\infty} x f_X(x)dx$$

If X is discrete with N possible values having possible values $P(x_i)$, then

$f_X(x) = \sum_{i=1}^{N} P(x_i)\delta(x - x_i)$ and $E[X] = \sum_{i=1}^{N} x_i P(x_i)$

Expected Value of a Function of a Random Variable

$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$ or, if X is discrete, $E[X] = \sum_{i=1}^{N} g(x_i) P(x_i)$

Moments

Moments about the origin have $g(X) = X^n \quad x = 0,1,2,3,\ldots$

The n^{th} moment about the origin is given by $m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$.

If $n = 1$, $m_1 = E[X] = \overline{X}$

Moments about the mean value of X are called central moments, denoted by μ_n. They are defined as the expected value of the function $g(X) = (X - \overline{X})^n \quad n = 0,1,2,\ldots$

$$\mu_n = E\left[(X - \overline{X})^n\right] = \int_{-\infty}^{\infty} (x - \overline{X})^n f_X(x) dx$$

$$\mu_0 = 1\text{, the area under } f_X(x) \qquad \mu_1 = E\left[(X - \overline{X})\right] = \int_{-\infty}^{\infty} (x - \overline{X}) f_X(x) dx = 0$$

Variance

Variance is defined as the second central moment and is denoted by σ_X^2.

$$\sigma_X^2 = \mu_2 = E\left[(X - \overline{X})^2\right] = \int_{-\infty}^{\infty} (x - \overline{X})^2 f_X(x) dx$$

σ_X is called the standard deviation of X.

$\sigma_X^2 = E\left[X^2 - 2X\overline{X} + \overline{X}^2\right] = E\left[X^2\right] - 2\overline{X}E[X] + \overline{X}^2 = E\left[X^2\right] - 2\overline{X}^2 + \overline{X}^2$

$= E\left[X^2\right] - \overline{X}^2 = m_2 - m_1^2$

Characteristic Function

The characteristic function of a r.v. X is defined by

$$\Phi_X(\omega) = E\left[e^{j\omega x}\right] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx \quad \text{where} \quad j = \sqrt{-1}$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} d\omega$$

$$m_n = (-j)^n \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$

Moment Generating Function

$$M_X(\nu) = E\left[e^{\nu x}\right] = \int_{-\infty}^{\infty} f_X(x) e^{\nu x} dx \quad \text{where} \quad -\infty < \nu < \infty$$

$$m_n = \left. \frac{d^n M_X(\nu)}{d\nu^n} \right|_{\nu=0}$$

Gaussian or Normal Random Variables

A random variable X is called Gaussian if its probability density function has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-m_x)^2}{2\sigma_X^2}}$$

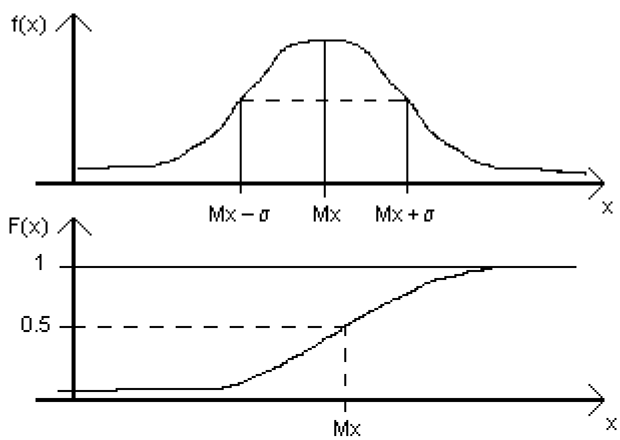
The density function has two parameters m_x and σ_X^2 where

$$m_x = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx \quad \text{and} \quad \sigma_X^2 = \int_{-\infty}^{\infty} (x - m_x)^2 f_X(x)dx$$

For shorthand notation we can write $X \sim N(m_x, \sigma_X^2)$

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} e^{-\frac{(\zeta - m_X)^2}{2\sigma_X^2}} d\zeta$$

This integral does not have a closed form solution and must be evaluated by numerical methods.



Binomial Distribution

If $0 < p < 1$ and $n = 1,2,3,\ldots$ then

$$f_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(x-k)$$

is called the binomial density function. The binomial distribution function is found:

$$F_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} u(x-k)$$

Exponential Distribution

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{x-a}{b}} & x > a \\ 0 & x < a \end{cases} \qquad F_X(x) = \begin{cases} 1 - e^{-\frac{x-a}{b}} & x > a \\ 0 & x < a \end{cases}$$

Examples

Example 1:

Using the exponential distribution:

$$E[X]=\int_a^\infty \frac{x}{b}e^{-\frac{x-a}{b}}dx=\frac{1}{b}e^{\frac{a}{b}}\int_a^\infty xe^{-\frac{x}{b}}dx=\frac{1}{b}e^{\frac{a}{b}}e^{\frac{x}{b}}\left(-bx-b^2\right)\Big|_a^\infty=a+b$$

Example 2:

A r.v. is Rayleigh distributed e.g., $f_X(x)=\begin{cases}\frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} & x>a\\ 0 & x<a\end{cases}$ with $a=0$, and $b=5$.

$$Y=g(V)=V^2$$

$$E[g(V)]=E\left[V^2\right]=\int_0^\infty \frac{2u^3}{5}e^{-\frac{u^2}{5}}du$$

Let $\zeta=\frac{u^2}{5}$, so $d\zeta=\frac{2}{5}udu$ and $E[g(V)]=5\int_0^\infty \zeta e^{-\zeta}d\zeta=5$

Example 3:

Using the exponential distribution:

$\sigma_X^2=\int_a^\infty \left(x-\overline{X}\right)^2\frac{1}{b}e^{-\frac{x-a}{b}}dx$ Now let $u=x-\overline{X}$.

$$\sigma_X^2=\frac{1}{b}e^{-\frac{\overline{X}-a}{b}}\int_{a-\overline{X}}^\infty u^2e^{-\frac{u}{b}}du=\left(a+b-\overline{X}\right)^2+b^2$$

From Example 1, $\overline{X}=a+b$, so $\sigma_X^2=b^2$

Example 4:

Using the exponential distribution:

$$\Phi_X(\omega)=\int_a^\infty \frac{1}{b}e^{-\frac{x-a}{b}}e^{j\omega x}dx=\frac{e^{\frac{a}{b}}}{b}\int_a^\infty e^{-\left(\frac{1}{b}-j\omega\right)x}dx=\frac{1}{b}e^{\frac{a}{b}}\frac{e^{-\left(\frac{1}{b}-j\omega\right)x}}{-\left(\frac{1}{b}-j\omega\right)}\Bigg|_a^\infty=\frac{e^{j\omega a}}{1-j\omega b}$$

$$\frac{d\Phi_X(x)}{d\omega}=e^{j\omega a}\left[\frac{ja}{1-j\omega b}+\frac{jb}{(1-j\omega b)^2}\right]$$

$$m_1=(-j)\frac{d\Phi_X(\omega)}{d\omega}\Bigg|_{\omega=0}=a+b$$