EE 525
Notes on Joint Random Variables
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Joint Distribution

In the case of one random variable, event *A* is defined by $A = \{X \le x\}$. A similar event *B* can be defined for *Y*, $B = \{Y \le y\}$. The probability distribution functions are $F_X(x) = P\{X \le x\}$ and $F_Y(y) = P\{Y \le y\}$. The joint event is defined by $\{X \le x, Y \le y\}$. The joint probability distribution function is given by $F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$. It is obvious that $F_{X,Y}(x,y) = P\{X \le x, Y \le y\} = P(A \cap B)$. For *N* r.v.'s $X_n = 1,2,\ldots,N$ the joint distribution function, denoted by $F_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N)$ is defined as the probability of the event $\{X_1 \le x_1, X_2 \le x_2, \ldots, X_N \le x_N\}$. $F_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N) = P\{X_1 \le x_1, X_2 \le x_2, \ldots, X_N \le x_N\}$

Properties of the Joint Distribution

1) $F_{X,Y}(-\infty,-\infty) = F_{X,Y}(-\infty,y) = F_{X,Y}(x,-\infty) = 0$
2) $F_{X,Y}(\infty,\infty) = 1$
3) $0 \le F_{X,Y}(x,y) \le 1$
4) $F_{X,Y}(x,y)$ is a non-decreasing function of *x* and *y*
5) $P\{x_1 < x \le x_2, y_1 < y \le y_2\} = F_{X,Y}(x_1,y_1) + F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1)$
6) $F_{X,Y}(x,\infty) = F_X(x)$, $F_{X,Y}(\infty,y) = F_Y(y)$

Property 6 defines the marginal distribution function.

Proof:

$P\{X \le x, Y \le y\} = P(A \cap B)$

If $y = \infty$, this implies $B = \{Y < \infty\} = S$

i.e., $F_{X,Y}(x,\infty) = P(A \cap S) = P(A) = P\{X \le x\} = F_X(x)$

Joint Density Function

For two random variables *X* and *Y*, the joint probability density function, denoted by $f_{X,Y}(x,y)$, is defined by

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

For *N* random variables

$$f_{X_1,X_2,\ldots,X_N} = \frac{\partial^N F_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N)}{\partial x_1 \partial x_2 \ldots \partial x_N}$$

$$F_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N) = \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_{X_1,X_2,\ldots,X_N}(\zeta_1,\zeta_2,\ldots,\zeta_N)\, d\zeta_1 d\zeta_2 \ldots d\zeta_N$$

Properties of the Joint Density

1) $f_{X,Y}(x,y) \geq 0$

2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$

3) $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2$

4) $F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2$

5) $P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)dxdy$

6) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$

Marginal Density Function

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad f_Y(y) = \frac{dF_Y(y)}{dy}$$

Conditional Distribution

$$F_X(x|B) = P\{X \leq x|B\} = \frac{P\{X \leq x \cap B\}}{P(B)} \qquad f_X(x|B) = \frac{dF(x|B)}{dx}$$

It can be shown

$$f_X(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Statistical Independence

Recall that two events are statistically independent iff $P(A \cap B) = P(A)P(B)$. This can be extended to two random variables *X* and *Y*. Let $A = \{X \leq x\}$ and $B = \{Y \leq y\}$. Two random variables are independent iff $P\{X \leq x, Y \leq y\} = P\{X \leq x\}P\{Y \leq y\}$. This implies

$$F_{X,Y}(x,y) = F_X(x)F_Y(y) \text{ and } f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

The conditional distribution for independent variables:

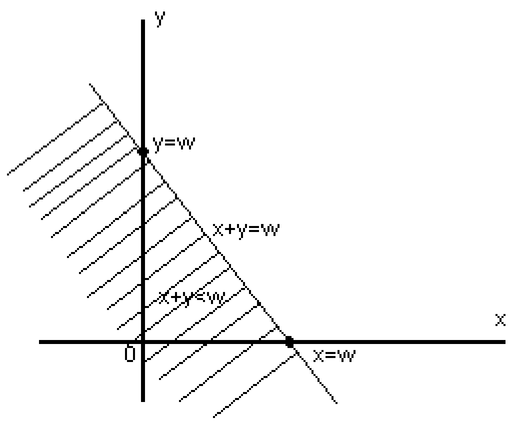
$$F_X(x|Y \leq y) = \frac{P\{X \leq x, Y \leq y\}}{P\{Y \leq y\}} = \frac{F_{X,Y}(x,y)}{F_Y(y)} = F_X(x)$$

Similarly

$$F_Y(y|X \leq x) = F_Y(y) \quad f_X(x|Y \leq y) = f_X(x) \quad f_Y(y|X \leq x) = f_Y(y)$$

Sum of Two Random Variables

Let $W = X + Y$ where X and Y are random variables. $F_W(w) = P\{W \le w\} = P\{X + Y \le w\}$



The probability corresponding to an elemental area *dxdy* in the *xy* plane is $f_{X,Y}(x, y)dxdy$