Introduction to
Nyquist Plots & Stability Criteria

EE302
DePiero
Nyquist Criteria Useful

- Determine Stability
- Determine Gain & Phase Margins
- ‘Medium’ effort. Finds number of RHP poles of $T(s)$, the closed-loop transfer function.
- Does not find pole values explicitly. (Similar to with Routh-Hurwitz).
Define $F(s) = \text{Denominator of } T(s)$

- $T(s) = \frac{KG(s)}{1 + KGH(s)}$
- $F(s) = 1 + KGH(s)$
- $T(s)$ is stable iff zeros of $F(s)$ are in LHP.
- Note:
  
  Zeros of $F(s)$ are _____ of $T(s)$, which are hard/easy to find.

  Poles of $F(s)$ are _____ of $KGH(s)$, which are hard/easy to find.
Consider Contour in S-Plane

- Contour travels up jw axis.
- Contour encircles all RHP poles/zeros.
- $|R| \to \infty$.
- Assume for now: No poles on jw axis(*)

* Integration along contour must avoid infinite values (rude)
Consider Net Phase Change To Factors of F(s)

\[ F(s) = \frac{(s + z_1) \cdots (s + z_M)}{(s + p_1) \cdots (s + p_N)} \]

- Point \( s = s_0 \) moves around contour CW direction.
- Vector differences \((s_0+p_j)\) experience change in phase angle.
- What is the accumulated phase contribution to \(<F(s)\) from \(<(s_0+p_j)\), as \(s_0\) traverses contour?

Imagine \((s_0+p_j)\) to be a handle of a crank winding a spring…
Integration of phase \(<(s_0+p_j)\) along contour analogous to winding spring

<table>
<thead>
<tr>
<th>???</th>
<th>RHP</th>
<th>LHP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero</td>
<td></td>
<td></td>
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<tr>
<td>Pole</td>
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</table>
Number of RHP Poles & Zeros Are Revealed by Net Phase Change

- Define
  
  \[ Z = \# \text{ RHP Zeros of } F(s) = \# \text{ RHP Poles of } T(s) \quad [\text{Something we want to know}] \]
  
  \[ P = \# \text{ RHP Poles of } F(s) = \# \text{ RHP Poles of } KGH(s) \quad [\text{Something easy to find}] \]
  
  \[ N = (\text{Net phase change in } F(s) \text{ as } s \text{ traverses contour CW}) \]
  
  -360 Degrees

- Example
  
  - If \( P=0 \), \( Z=1 \) Then \( N = \) ___
  - If \( P=1 \), \( Z=0 \) Then \( N = \) ___
  - If \( P=1 \), \( Z=1 \) Then \( N = \) ___

- Relation Between \( N, Z, P ? \) __\( Z = \) ________
New Representation: F(s)-Plane

- Plot F(s) as s varies along contour.
- Phase of F(s=s0) directly observable from plot.
  
  Consider polar form of F(s), a ‘polar plot’.

- Accumulated phase change of F(s) directly observable from plot. What is the criteria for N=1? 

  \[ N = (\text{Net phase change in } F(s) \text{ as } s \text{ traverses contour CW}) \]

  -360 Degrees

Imagine F(s=s0) to be a handle of a crank winding a spring…
F(s)-Plane Representation
Useful to Find Net Phase Change, N

- Define $N =$ Number of CW encirclements of origin, in the F(s)-Plane Plot.

Also note $N$ can be negative – corresponding to CCW encirclements.
KGH(s)-Plane More Convenient To Determine Stability

- Note: $\text{KGH}(s) = F(s) - 1$.
- Hence plot of $\text{KGH}(s)$ is shifted version of $F(s)$-Plane plot.
- $N$ is determined by number of CW encirclements of $-1$.
- Nyquist Stability Theorem (Formally stated)
  - If $P=0$ then stable iff no encirclements of $-1$.
  - If $P\neq0$ then stable iff $Z = P + N = 0$
- Procedure:
  1. Find the $\text{KGH}(s)$-Plot
  2. Examine plot, find $N$
  3. Examine factors of $\text{KGH}(s)$, to find $P =$ # RHP Poles of $\text{KGH}(s)$
  4. $Z = P + N$, $Z =$ #RHP Poles of $T(s)$
  5. Stable iff $Z = 0$

Note: Factors of $\text{KGH}(s)$ typically easy to find, as open loop transfer function is usually built up from several cascaded (simpler) blocks.
Stability Test Easy, Given Plot

\[ KGH(s) = \frac{6}{(s+1)(s+2)(s+3)} \]

- \( N = \) 
  # CW encirclements of -1
- \( P = \) 
  # RHP poles of \( KGH(s) \)
- \( Z = P + N = \) 
  # RHP poles of \( T(s) \)
- Stable? Yes/No?

herringbone shows increasing \( jw \) (Green \( jw < 0 \))
Shape of Nyquist Plot Specific to Gain (K), Reveals Stability

\[ KGH(s) = \frac{K}{(s+1)(s+2)(s+3)} \]

- With K>60 the Nyquist Plot encircles \(-1\) point in the CW direction.
- (Alt Approaches: RL+Routh or Bode)
Gain Margin Found Easily Given KGH(s) Plot

- Increasing K scales plot linearly (enlarges).
- Nyquist contour intersects real axis at $-0.5$ (for $K=30$). Doubling $K$ causes instability.
- Confirmed with Bode plot ($2x = +6\text{dB}$).
Phase Margin Found Easily
Given KGH(s) Plot

- Increasing phase lag in open loop system equivalent to CW rotation of F(s) plot.
- Phase Margin found by determining rotation needed for contour to intersect –1 point.
- Confirmed with Bode (previous) as 26 deg.
Exclude Poles/Zeros on jw Axis Except for Integrators

- Can’t integrate over a pole – yields infinite (rude) result.
- Adjust contour in s-plane to move around poles and zeros. Use tiny radius.
- Exclusion eliminates contribution to net phase change of F(s).
- Typically not effecting # of encirclements of -1 point.
- Omitting cases with KGH(s) having poles on jw axis, other than origin…
Summary: Learning Objectives

• Construct a Nyquist plot given KGH(s).
• Determine stability using a Nyquist plot:
  1. Find the KGH(s)-Plot
  2. Examine plot, find $N = \#$ CW encirclements of -1
  3. Examine factors of KGH(s), to find $P = \#$ RHP Poles
  4. $Z = P + N$
• Nyquist Stability Theorem (Formally stated)
  – If $P=0$ then stable iff no encirclements of -1.
  – If $P!=0$ then stable iff $Z = P + N = 0$
• Find Gain/Phase Margins given Nyquist plot.
  – GM: Increase in $K$ necessary to scale plot to encircle $-1$.
  – PM: Rotation of plot CW needed to encircle $-1$. 
Consider Limits When Plotting KGH(s)

\[ KGH(s) = \frac{6}{(s + 1)(s + 2)(s + 3)} \]

- Note symmetry above & below real axis.
- Contribution to plot for \(|jw| \rightarrow \infty\) collapses to a single point.

<table>
<thead>
<tr>
<th>Limit</th>
<th>KGH(s)=?</th>
</tr>
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<tbody>
<tr>
<td>(jw \rightarrow +0)</td>
<td></td>
</tr>
<tr>
<td>(jw \rightarrow -0)</td>
<td></td>
</tr>
<tr>
<td>(jw \rightarrow +\infty)</td>
<td></td>
</tr>
<tr>
<td>(jw \rightarrow -\infty)</td>
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