Compensator networks are (typically) implemented as electrical circuits in the feed-forward path of a control system. Compensators may be employed to improve system performance and can permit an increased forward gain (K) to reduce steady-state error, for example. Another use of compensators is to improve damping and thus reduce overshoot and improve settling time.

This handout addresses phase lead compensators and improved damping, specifically. Phase lead compensators are placed in cascade, in the forward path, and are used to manipulate the phase margin. Phase lead compensators shift Bode phase plots of KGH(jw) (open loop transfer function) upward in a desired frequency range. This increases phase margin, which improves damping, as described below. Fundamentally a phase lead network is beneficial because often other (electromechanical) components introduce phase lag, due to inertia and inductance. The phase lead network counters these plant dynamics. Improvements from compensation are illustrated in Figure 1.

Figure 1. Bode plots (above) before and after compensation (l. to r.) and corresponding step responses (below). The magnitude curve on the Bode plots is darker and phase curve lighter. Improved damping of the step response is evident in the lower right, with reduced overshoot and reduced settling time.
Adjusting phase margin to achieve a given damping ratio is relatively easy because (an approximate) linear relationship exists: \( \phi_{pm} = 100^\circ \xi \) (degrees). See Figure 2.

![Graph showing linear approximation between damping ratio and phase margin.](image)

**Figure 2.** An approximate, linear relationship exists between damping ratio and phase margin. This is useful when selecting new phase margins in compensator design. From “Modern Control Systems” by Dorf.

A phase lead compensator has a transfer function

\[
G_c(s) = \frac{1 + \alpha \tau s}{\alpha(1 + \tau s)} = \frac{s + z}{s + p}
\]

Note the attenuation by \( 1/\alpha \). This accounted for by increasing the forward gain (K) by a factor of \( \alpha \). The compensators \( \alpha \) and \( \tau \) are determined via the following procedure.

1. Make a Bode plot of \( KGH(jw) \), including log-magnitude and phase plots.
2. Find the phase margin of the initial (uncompensated) system.
3. Select a new, desired, phase margin. This may be based on given requirements (such as a given damping ratio).
4. Find \( \phi_m = \text{desired phase margin} - \text{initial phase margin} + \text{increased margin} \). An increase is a good idea due to approximations in design equations and due to component variations in real systems. The phase of the compensator network varies with frequency (see Figure 3) and \( \phi_m \) is the maximum phase value.
5. Find \( \alpha \) via \( \sin(\phi_m) = \frac{\alpha - 1}{\alpha + 1} \), which defines the separation between the pole and zero of the compensator network.
6. Find \( w_m \). This is the frequency that the compensator provides its maximum phase contribution. This maximum contribution should directly affect the new phase margin. Hence \( w_m \) is chosen to correspond to the 0 dB-crossing frequency of the compensated system. This corresponds to the frequency associated with a magnitude of \(-10 \log \alpha\) in the uncompensated system.
7. Find \( G_c(s) \). Pole = \(-p = 1/\tau = w_m \sqrt{\alpha} \). Zero = \(-z = -p/\alpha = 1/\tau \alpha\)
8. Find circuit values via \( \tau = \frac{R_1R_2}{R_1 + R_2}C \) and \( \alpha = \frac{R_1 + R_2}{R_1R_2} \). See Figure 3.
9. Boost forward gain (K) by \( \alpha \).
Figure 3. Electrical network for the phase lead compensator (below). Bode plot of magnitude and phase characteristics of the compensator (above). The electrical network must be followed by a gain stage of $\alpha$ (to match the Bode plot). The maximum phase contribution of the compensator $\phi_m$ occurs at the frequency $\omega_m$. This is at the geometric mean of the pole and zero frequencies. Also from Dorf text.

The effect of a compensator may be appreciated in several different ways.

- **Phase Change / Time Delay Perspective**: As the phase lag increases around the open loop, it can potentially reach $180^\circ$. Such a phase change, together with the normal negation of a feedback control system amounts to positive feedback and an unstable response. Adding a phase lead filter in the forward path reduces the accumulated phase. Note that even though the open loop phase change reaches $180^\circ$ for only one frequency, real signals typically contain (at least a bit of) energy at all frequencies. Any nonzero amount would begin the positive feedback process, causing the output to grow without bound.

- **Bode Perspective**: Given that phase margin is linearly related to damping, the phase margin is readily adjusted by adding a compensation block to the forward path. This shifts the phase margin, improving damping.

- **Root Locus Perspective**: When poles or zeros are added to the forward path these shift the root locus curves – by changing the number of asymptotes or asymptote center. Hence adding a compensator can redirect the locus to regions of the S-plane with greater damping, for example. See Figure 4.
Figure 4. Root locus curves before and after compensation (l. to r.). The compensator adds a real pole and zero. This shifts the curves away from the jw axis, increasing the amount of damping that may be achieved. Closed loop pole locations of the more dominant poles are indicated for gain values of K=30 and K=90.

As indicated in Figure 4, compensation can provide additional benefits, by reducing sensitivity. The closed loop pole locations are shown in Figure 4, for K=30 and K=90. This section of the locus is approximately a straight line, with a nearly constant damping ratio. Hence the system maintains a near constant amount of damping, across a relatively wide range of gain. Figure 5 shows the step response over this range of gain values. Note the rapid settling compared to the uncompensated system in Figure 1.

Figure 5. Step response associated with varying gain values: K=30, 50, 70, 90 are shown in (a,b,c,d) respectively. Rapid settling is obtained despite variations in gain, compared to the uncompensated system.