2/10/03

1. Review, grade HW #4 in class.
2. HW #5 due Wed 2/19 4:10, 4:11.
3. Review of HW #3, and in particular details.
4. Lecture:

MAP \Rightarrow \max P(H_c | z_1, \ldots, z_K) = \frac{f(z_1, \ldots, z_K | H_c)}{f(z_1, \ldots, z_K)}

\Rightarrow C \cdot f(z_1, \ldots, z_K | H_c) \cdot P(H_c)

\text{H}_0: (z_1, \ldots, z_K) \text{unrelated} \rightarrow \text{independent gaussian rv's with } E[z_k] = \mu_k = (\mu_1, \mu_2)

\text{and } E[z_k \cdot z_{k'}] = \sigma_k^2 \delta_{kk'}

\Rightarrow f(z_1, \ldots, z_K | H_0) = \frac{1}{(2\pi \sigma_k^2)^{K/2}} \exp\left(-\frac{1}{2\sigma_k^2} \sum\sigma_k^{-2} \|z_k - \mu_k\|^2\right)

\text{with } \sigma_k = \sqrt{\frac{\sum_{k=1}^{K} \sigma_k^2 \phi_k(\alpha_k)}}{1 - \alpha_k}, \text{ for } t 

\sigma_k \equiv \Delta(\alpha_k) = \sqrt{\frac{\sum_{k=1}^{K} \sigma_k^2 \phi_k(\alpha_k)}}{1 - \alpha_k}, \text{ for } t 

\text{maximize } \ln P(H_c | z_1, \ldots, z_K) \Rightarrow \min \|z_k - \mu_k\|^2 = \frac{1}{\sigma_k^2} \sum\sigma_k^2 \|z_k - \mu_k\|^2

\Rightarrow \min \|z_k - \mu_k\|^2 - \frac{1}{\sigma_k^2} \sum\sigma_k^2 \|z_k - \mu_k\|^2

\delta \equiv \max \left\{ (x-2)^2 + k \right\} \rightarrow \min \left\{ (x-2)^2 - 1 \right\}

\text{now if } P(\hat{H}_c) \text{ equally likely } \Rightarrow \text{probable } \Rightarrow \text{minimize } \|z_k - \mu_k\|^2 = \sum\|z_k - \mu_k\|^2

\text{by Parseval's Theorem } \Rightarrow \text{can determine decision regions for } K \text{-dimensional space } R^K.

\text{Calculation of probability of error for MAP - estimate based upon pairs}

P_2(\alpha) = 1 - P_3(\alpha) = 1 - \sum_{k=1} P(\hat{H}_c | H_k) P(H_k)

\text{Symbol error with } P_3(\alpha) = \text{correct reception of symbol } i \text{ averaged over all signals}

P(\hat{H}_c | H_k) \equiv \text{correct reception given the (i.e., } \alpha_k \text{) sent}

\text{with decision region } R_2

P(\hat{H}_c | H_k) = \int_{R_k} \exp\left(-\frac{1}{2\sigma_k^2} \|z_k - \mu_k\|^2\right) \frac{1}{(2\pi)^{K/2}} \exp\left(-\frac{1}{2\sigma_k^2} \|z_k - \mu_k\|^2\right) \text{ d}z

\text{Weinberg and Jacobs observation (ref. 2)}

1. Rotation and translation of \( \{ \phi_k(\alpha_k) \} \) do not affect \( P(\hat{H}_c | H_k) \)

2. Peak and average energies of signals to achieve \( P(\hat{H}_c) \) can be minimized by translation \( z = \sum_{k=1}^K \frac{1}{\alpha_k} P(H_k) \phi_k \) (see fig. 4-5)
3. Since noise $N_a$ is independent g.r.u.i. with $E N_a = 0$ and $E N_a N_e = -N_0 S_a q$, $P_e = \frac{1}{2}$ decisions on each $e_2$ $(Z_k)$ is independent of other $2^n$ decisions.

A. If $S_{a,i}$ are completely symmetric, then $P_e(e)$ is independent of $P(C_i)$ see Fig. 4-3.

New estimate of $P_e(e)$:

$$P_{e} (e | H_a) = P \left( \bigcup_{i=1}^{N_a} e_i \bigg| H_a \right)$$

where $e_i$ is event $S_i$ decided when $S_a$ sent $e_i$.

Use union bound for "dominant pair" $P(A \cup B) \leq P(A) + P(B)$

$$P_{e} (e | H_a) \leq \sum_{i=1}^{N_a} P(S_i, S_a) \text{ where } P(S_i, S_a) = P(e_i | e_i)$$

Choose dominant pair, let $d = || S_i - S_a ||$ and direction from $S_a$ to $S_i$.

$$\phi_i = \frac{S_i - S_a}{|| S_i - S_a ||} \text{ with } \mu_i = (n(t), \phi_i(t))$$

$$= \frac{\exp\left(\frac{-u^2}{2(N_0)^2}\right)}{\sqrt{2\pi (N_0)^2}} \mathrm{du} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \tau}} \exp\left(\frac{-v^2}{2\tau}\right) \mathrm{dv}$$

$$= F \left( \frac{d}{\sqrt{N_0}} \right)$$

$$= Q \left( \frac{|| S_i - S_a ||}{\sqrt{2N_0}} \right)$$

$$P_e(e) \approx Q \left( \frac{|| S_i - S_a ||}{\sqrt{2N_0}} \right)$$

Where $S_i$ is the "closest" $S_i$ to $S_a$ in signal space.
Review for midterm exam

(see fig 1-3)

Shannon's Paper:

Thm 1. Shannon Sampling Theorem, and EUT [Section 2.4]

Thm 2. C = W log2 (1 + P/N) bits/s [Section 1.2, 2.3], P < C \Rightarrow P_e \to 0

Thm 3. Source Encoding [Section 3.2] Huffman

Power line calculations: white noise [2.17], initial PSD [2.18]

Binary Channel (Chapter 3)

(see Fig. 3-1) min P_e \Rightarrow use of Q, water function

matched filter \Rightarrow max S^2 = \min P_e , \text{ correlation receiver }

(see Fig. 3-7) Nyquist pulse shaping criterion for Bu = W channel

Pr(e) \to 0 \Rightarrow \text{ perfect match} \Rightarrow P_e = \frac{2B \sigma^2}{N_0}

Examples 3-6, 3-7

Signal Space (Chapter 4)

(see Fig. 4-1) Rates: messages, symbols, bits/s

MAP criterion \Leftrightarrow \min P_e \Leftrightarrow \max S^2 \quad \text{Example 4-1}

Hilbert Space model

f(x) \in L^2, \quad (f,g) = \int f(x)g^*(x)dx

Complete space: all Cauchy sequences \to C(0, x)

Gram-Schmidt: o.n. basis for K \leq M signals

MAP receive \Leftrightarrow \text{matched filter} \Rightarrow \text{correlation}

P(e) = Q\left(\frac{d}{\sqrt{N_0}}\right) \quad \text{with} \quad d = \text{SNR of signal sent}